

Spinor Bose gas in an elongated trap

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We examine a spinor Bose gas confined by an elongated trap. Since a spin-independent energy is much higher than a spin-dependent energy in alkali species, the system exhibits different properties by changing a radial confinement. We show that if a spin-dependent coupling is positive, a spin-liquid condensate, which breaks the charge $U(1)$ symmetry but preserves the spin rotational symmetry, can be realized in an intermediate confinement regime. Properties of the spin-liquid condensate are visible if a temperature is lower than a spin gap to characterize the spin-disorder property. If a temperature is higher than the gap but lower than a spin-dependent coupling energy, on the other hand, a regime in which a spin sector is described by a semiclassical wave emerges. A characterization in each regime by means of correlation functions and topological solitons is also discussed.

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I. INTRODUCTION

Quantum many-body physics in reduced dimension is fundamentally different from that in higher dimensions [1]. When it comes to bosonic systems, whilst the presence of a Bose-Einstein condensate (BEC) [2] is the natural consequence for the systems in higher dimensions, there is no BEC in an interacting one dimensional system due to quantum fluctuations [3, 4]. Instead in many cases the so-called Tomonaga-Luttinger liquid (TLL) [1], which is also responsible to describe many fermionic one dimensional systems, is realized.

A well-known criterion for the emergence of quantum one dimensional systems is that temperature and interaction energy should be much smaller than a confinement energy along higher dimensional directions [1]. However, one may encounter a nontrivial low-dimensional system if there are several energy scales each of which is energetically separated. A typical example is a quasi-one dimensional superconductor discussed in condensed matter physics where an energy scale of electrons is much higher than that of Cooper pairs. In this system, therefore, it is expected that by changing a radial confinement properties of the superconductor change despite the three dimensional motion in each electron [5, 6].

Recently, on the other hand, cold atoms have provided remarkable realizations of systems in reduced dimension [7–9]. Practically speaking, such systems can be prepared by making a trap very anisotropic [10–12] or by loading systems on two-dimensional optical lattices [12–14].

In particular, one can consider a spinor Bose gas realized in cold atoms [15, 16] as a system with multiple energy scales. Experimentally, this system has been realized with alkali species such as ^{23}Na and ^{87}Rb where s -wave scattering lengths to characterize interatomic interactions take similar values for different spin channels. This implies that a spin-dependent coupling is much smaller than a spin-independent coupling. Therefore, a variance of spin can easily be restricted to a one direction compared with that of charge, which has indeed been confirmed in Ref. [17].

In this paper, we examine a spinor Bose gas confined by an elongated trap as illustrated in Fig. 1 and show that the system experiences nontrivial properties by changing a radial confine-



FIG. 1. (color online) Spinor Bose gas in an elongated trap with a cross section s .

ment. In particular, by considering a positive spin-dependent coupling, we predict that in a region where a charge sector is three dimensional while a spin sector is one dimensional, a spin-liquid condensate [18–22] is realized. We also discuss temperature effects and show that a regime where the spin sector is described by a semiclassical wave [23–26] is achieved in certain temperature regime although it is difficult to obtain such a regime in an antiferromagnetic quantum spin-1 system [23]. This difference can be attributed to the fact that there are several length scales in spinor bosons while there is only one length scale in a quantum spin system. A characterization in each phase by means of correlation functions and vortices is also discussed.

II. MODEL

We consider spin-1 bosons in an elongated trap. The Hamiltonian with $U(1) \times SO(3)$ symmetry is given by

$$H = \int d^3x \left[\psi_m^\dagger \left(-\frac{\hbar^2 \Delta}{2M} + V(\mathbf{x}) \right) \psi_m + \frac{c_0}{2} (\psi_m^\dagger \psi_m)^2 + \frac{c_1}{2} (\psi_m^\dagger \mathbf{F}_{mn} \psi_n)^2 \right], \quad (1)$$

where ψ_m is the boson field with the mass M , F_i ($i = x, y, z$) are spin-1 matrices, and $V(\mathbf{x})$ is a confined potential described by a harmonic oscillator. In what follows, let us focus on a situation that a confined potential is imposed only along the radial direction, namely, $V(\mathbf{x}) = \frac{M\omega_r^2}{2}(y^2 + z^2)$. In fact, such a trapping has already been realized in Ref. [27]. The spin-independent and spin-dependent couplings are respectively expressed with s -wave scattering lengths as

$c_0 = 4\pi\hbar^2(a_0 + 2a_2)/3M$ and $c_1 = 4\pi\hbar^2(a_2 - a_0)/3M$, where a_f ($f = 0, 2$) is the s -wave scattering length of the total spin- f channel.

III. SPINOR BOSE GAS IN AN ELONGATED TRAP

Here, we discuss a spin-1 Bose gas for $c_1 > 0$ in different harmonic oscillator frequencies at absolute zero.

A. 3D regime

We consider a regime obeying the following condition:

$$nc_0, nc_1 \gg \hbar\omega_\perp \quad (2)$$

where n is the density, nc_0 and nc_1 denote the interaction entries with respect to charge and spin, respectively. In this regime, since there are a number of transverse modes, the system is treated as 3D and thereby the Gross-Pitaevskii approximation is justified at least for a weak-coupling case. This approximation is nothing but the substitution of a c -number field ϕ_m for the original field. Such a c -number field can be determined by the saddle point equation known as the Gross-Pitaevskii equation. It can be shown that for $c_1 > 0$ the ground state is a polar phase whose order parameter is given by [28, 29]

$$\langle\psi_m\rangle \equiv \phi_m = \sqrt{n}(0, 1, 0)^T, \quad (3)$$

where T represents transposition. We also note that in general n has a dependence of r_\perp , which may be approximated by the Thomas-Fermi profile due to the condition (2) [2].

Let us next look at a symmetry breaking. From (3), we see that $\langle\mathbf{F}\rangle = \mathbf{0}$ and the charge $U(1)$ symmetry is spontaneously broken, which is accompanied by an off-diagonal long-range order (ODLRO). In addition, it is easy to show [30]

$$\langle[\psi_{m'}, \psi_m^\dagger(F_i)_{mm'}]\rangle \neq 0, \quad (i = x, y) \quad (4)$$

which states that the spin rotational symmetry is also spontaneously broken as $SO(3) \rightarrow SO(2)$. An accurate order parameter manifold including discrete symmetry can be obtained by operating $e^{i\theta}e^{iF_z\alpha}e^{iF_y\beta}e^{iF_x\gamma}$ into Eq. (3), where θ is the phase of the charge $U(1)$ and (α, β, γ) are Euler angles, and is shown to be [31]

$$G/H = [U(1) \times S^2]/Z_2. \quad (5)$$

The above order parameter manifold physically represents that while the charge $U(1)$ symmetry and rotational symmetry along the x and y axes are broken, the rotational symmetry along the z and spin-charge coupled discrete Z_2 symmetry remain.

Due to the spontaneous symmetry breaking, gapless modes known to be Nambu-Goldstone modes emerge. The number of Nambu-Goldstone modes is expected to be equal to the dimension of the order parameter manifold, which is 3 in Eq. (5). In fact, we can check this forecast by means of the analysis of collective modes. To this end, we consider fluctuation

effects from the c -number field and expand it up to the linear order of the fluctuation field in the Gross-Pitaevskii equation, which leads to the Bogoliubov equation. By diagonalizing it, we obtain the following Bogoliubov modes: [28, 29]

$$E_{c,\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2nc_0)}, \quad (6)$$

$$E_{f,i,\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2nc_1)}, \quad (i = x, y) \quad (7)$$

where $\epsilon_{\mathbf{k}} = \hbar^2 k^2/(2M)$, $E_{c,\mathbf{k}}$ denotes the Bogoliubov mode coming from charge fluctuations, and $E_{f,x,\mathbf{k}}$ and $E_{f,y,\mathbf{k}}$ are related to spin fluctuations. Thus, it turns out that all of the above Bogoliubov modes are shown to be linear gapless by taking the limit $k \rightarrow 0$, which is consistent with the Nambu-Goldstone theorem.

B. 1D regime

We next consider a situation that atomic motions along the transverse directions are frozen and the gas is kinematically 1D, which implies

$$nc_0, nc_1 \ll \hbar\omega_\perp. \quad (8)$$

In this case, the conventional Gross-Pitaevskii approach is no longer appropriate due to the absence of a BEC. Nevertheless, we can employ a semiclassical analysis to describe low-energy properties in the system.

A key idea is an introduction of a number-phase representation in bosons [32],

$$\psi_m = \sqrt{nn_m}e^{i\theta_m}, \quad (9)$$

with a constraint $\sum_m n_m^2 = 1$. If we discuss such a representation in spinless bosons, there are two variables, which are density and phase. Since the density is a massive degree of freedom, we can integrate out it as far as low-energy properties in the system are concerned. As a consequence, we obtain a phase-only action, which is essentially the TLL whose spectrum is linear gapless. This is indeed the correct low-energy effective action [1]. For the case of spinor bosons, on the other hand while the situation is much involved due to the multiple number-phase variables, the technique used in spinless bosons is still applicable. First, by solving saddle point equations in the number-phase representation, we can specify massive and massless degrees of freedom at the lowest order. Second, by expanding the effective potential from the minimum with respect to the massive degrees of freedom up to second order and then integrating out them, we arrive at the following effective action [31, 32]:

$$S = S_{TLL} + S_{NL\sigma M}, \quad (10)$$

$$S_{TLL} = \frac{\hbar K_c}{2\pi} \int dt dx \left[\frac{1}{v_c} (\partial_t \theta_+)^2 - v_c (\partial_x \theta_+)^2 \right], \quad (11)$$

$$S_{NL\sigma M} = \frac{\hbar}{2g} \int v_s dt dx \left[(\partial_t \mathbf{m}/v_s)^2 - \{(\partial_x \mathbf{m})^2\} \right]. \quad (12)$$

Here S_{TLL} is the TLL action reflecting in a superfluid property, which can be described by the velocity $v_c = \sqrt{\hbar c_0/M}$

and the so-called TLL parameter $K_c = \pi\hbar\sqrt{\bar{n}/(M\bar{c}_0)}$ with the line density $\bar{n} = ns$ and effective 1D coupling $\bar{c}_i = c_i/s$ [33]. On the other hand, $S_{NL\sigma M}$ is the nonlinear σ model action describing the spin dynamics with velocity $v_s = \sqrt{\hbar\bar{c}_1/M}$, dimensionless coupling constant $g = \sqrt{M\bar{c}_1/(\hbar^2\bar{n})}$, and constraint $\mathbf{m}^2 = 1$. The variables in the above action are associated with (9) via $\theta_{\pm} = (\theta_1 \pm \theta_{-1})/2$, and $\mathbf{m} = (\sin\theta\cos\theta_{-}, \sin\theta\sin\theta_{-}, \cos\theta)^T$ with $\theta = \sin^{-1}[(n_1 - n_{-1})/\sqrt{2}]$. Thus, we could obtain the action with the spin-charge separation, which is naturally expected in one dimensional systems [1].

Here, we look at excitation properties in each sector. As for the charge sector, the gapless property of the spectrum should be maintained unless there is a commensurate potential, which introduces cosine terms and may make it gapful. As for the spin sector described by the 1+1 dimensional nonlinear σ model, the excitation is going to be gapped. This is in contrast with the two or three dimensional cases where the spin excitation is gapless due to the absence of infrared divergences [34]. In fact, the above properties can be shown exactly for the $c_0 = c_1$ case [35] where the so-called Takhtajan-Babujian limit [36, 37] is realized and the Bethe ansatz solution is available. In addition, the corresponding analysis shows that the ground state consists of a string solution forming a spin-singlet pair, which is responsible for the spin gap.

We now come back to the semiclassical analysis to see correlation properties. The above semiclassical analysis indicates that the bosonic field can be expressed as

$$\psi_m \approx \sqrt{n}e^{i\theta_+}((m_1 + im_2)/\sqrt{2}, m_3, -(m_1 - im_2)/\sqrt{2})^T. \quad (15)$$

Thus, we see that the one-particle correlation function in a long range decays exponentially since $\langle m_i(r)m_j(0) \rangle \sim \delta_{i,j}e^{-\Delta_s r/(\hbar v_s)}$, where Δ_s is the spin gap originating from the nonlinear σ model action (12), while a correlator of the charge sector decays algebraically as $\langle e^{i\theta_+(r)}e^{-i\theta_+(0)} \rangle \sim r^{-1/(2K_c)}$, where $r = \sqrt{x^2 + v_j^2 t^2}$ ($j = c$ or s). Thereby, the dominant correlation turns out to be the following spin-singlet pair correlation:

$$\langle P_0^\dagger(r)P_0(0) \rangle \approx \langle e^{-2i\theta_+(r)}e^{2i\theta_+(0)} \rangle \sim r^{-2/K_c}, \quad (16)$$

where $P_0 = 2\psi_1\psi_{-1} - \psi_0\psi_0$. As a consequence, it follows that the ground-state of a one-dimensional spinor Bose gas for $c_1 > 0$ is the spin-liquid TLL.

C. Intermediate regime

In accord with the analyses discussed above, we next consider an intermediate confinement regime defined by

$$nc_1 \ll \hbar\omega_{\perp} \ll nc_0. \quad (17)$$

The above states that the charge sector is three dimensional while the spin sector is one dimensional, that is, in the latter sector a quantum fluctuation is important.

To consider this regime, we come back to the number-phase representation of bosons discussed in the one dimensional regime. An important observation is that the number-phase representation of Eq. (13) is still applicable for higher dimensional cases. Then, the crucial point is that for higher dimensions a quantum fluctuation effect trying to break the order is weak enough, which ensures that excitations both from the charge and spin are gapless, and leads to the ODLRO in the system. Thus, for higher dimensions Eq. (13) corresponds to (3) up to $U(1) \times SO(3)$ rotations.

On the other hand, from the condition (15), the spin correlation length $\xi_s = \hbar/\sqrt{2Mnc_1}$ providing a typical length scale in the spin sector is larger than the radial oscillator length $a_{\perp} = \sqrt{\hbar/M\omega_{\perp}}$ while the charge correlation length $\xi_c = \hbar/\sqrt{2Mnc_0}$ as a typical length scale in the charge sector is smaller than a_{\perp} . This implies that the effective low-energy action describing the intermediate regime is given by

$$S = S_c + S_{NL\sigma M}, \quad (18)$$

$$S_c = \hbar^2 \int dt d^3x \sqrt{\frac{n}{4Mc_0}} \left[\frac{1}{v_c} (\partial_t \theta_+)^2 - v_c (\nabla \theta_+)^2 \right], \quad (19)$$

where $S_{NL\sigma M}$ corresponds to Eq. (12). Since the effective action of the charge sector is three dimensional, a quantum fluctuation effect is significantly reduced to cause the emergence of a BEC. As in the case of the 3D regime, the density profile may be captured by the Thomas-Fermi one due to $nc_0 \gg \hbar\omega_{\perp}$. On the other hand, the effective action of the spin sector is the 1+1 dimensional nonlinear σ model and therefore, the spin-singlet formation and corresponding spin gap emerge.

Now, we look at correlation properties in this regime. A striking property is that the one-particle correlation function decays exponentially as in the case of the 1D regime, which implies the absence of an ODLRO of the one-particle density matrix. This is due to the fact that while $\langle e^{-i\theta_+(r)}e^{i\theta_+(0)} \rangle$ is a constant even in a long range, the one-particle correlation itself disappears in the corresponding limit since $\langle m_i(r)m_j(0) \rangle \sim \delta_{i,j}e^{-\Delta_s r/(\hbar v_s)}$. At the same time, this property does not mean the absence of an ODLRO in the system itself. In fact, the spin-singlet pair correlation function, $\langle P_0^\dagger(r)P_0(0) \rangle$ remains nonzero in a long range. Namely, as in the case of fermionic superfluids described by the BCS theory, the ODLRO comes out from the two-particle density matrix.

Here we point out that the BEC realized in this intermediate regime can be regarded as a spin-liquid condensate discussed in Refs. [18–22]. This is a bosonic state such that the spin rotational symmetry remains unbroken while the $U(1)$ charge symmetry breaks spontaneously. In our model, the $U(1)$ symmetry breaking causes the ODLRO of the two-particle density matrix not of the one-particle density matrix due to the spin-disorder property. We notice the difference between the spin-liquid condensate and the spin-singlet pair condensate obtained with a single-mode approximation [38–41]. In the latter case, while the spin rotational symmetry is maintained, the ODLRO of the one-particle density matrix exists since such a singlet-pair formation occurs between bosons with zero momentum. An emergence of bosonic condensates without the ODLRO of the one-particle density matrix is considered to be unusual.

IV. DISCUSSION

A. Characterization in each phase

Now, we wish to discuss a characterization in each phase in light of experimental observables. To this end, we focus on correlation properties and vortices, both of which can be measured in experiments. When it comes to correlation properties, the one-particle density matrix to confirm an ODLRO in a spinless BEC has been measured in [42, 43], and pair correlation function to confirm existence of pair condensation has been measured in [44] via the atom shot noise in absorption imaging. On the other hand, the several topological excitations including vortices in a spinor BEC have been reported in Refs. [45–52].

In the three dimensional regime (2), whose existence has been confirmed by the experiment [53], the ODLRO of the one-particle density matrix in the $m = 0$ component should appear in the order parameter (3). In addition, since the system size is bigger than the charge and spin correlation lengths, both of integer and half-quantized vortices are topologically allowed due to $H = Z_2 \ltimes U(1)$ and $\pi_1(G/H) = Z$ in the polar phase, where \ltimes and π_1 mean the semidirect product and first homotopy group, respectively.

In the one dimensional regime (8), vortex excitations are forbidden due to $a_\perp < \xi_c, \xi_s$. The one-particle density matrix decays exponentially in contrast with the three dimensional case. At the same time, since the charge part can be described by the TLL, we expect a quasi-long range order of the pair correlation function as Eq. (14).

In the intermediate regime (15), an integer quantum vortex is allowed although a half-quantum vortex is forbidden. This is due to the fact that there is (no) the rotational degree of freedom along the radial direction in the charge (spin) sector since $\xi_c < a_\perp < \xi_s$. The one-particle correlation decays exponentially as in the case of the one dimensional regime. On the other hand, the pair correlation function acquires the ODLRO because of the three dimensional properties in the charge sector.

B. Temperature effect

We turn to consider effects of a finite temperature within a range where the effective theory discussed above is applicable. This implies that the thermal lengths ($\hbar v_c/T$ and $\hbar v_s/T$) should be longer than the cut-off length of the theory in each sector, which is of the order of the healing lengths (ξ_c and ξ_s). In other words, we wish to consider the temperature satisfying $T < nc_0, nc_1$.

Although in the 3D regime we do not expect any modification of the argument at least in this temperature regime, we need to care about temperature effects in the one dimensional and intermediate regimes. This is because in both cases there is the spin gap Δ_s as an additional energy scale in the spin sector. The typical behavior of the spin gap is known to be $\Delta_s \sim nc_1 e^{-2\pi/g}$ in the weak-coupling limit, and is $\Delta_s \sim nc_1$

in the strong-coupling limit [54]. The dimensionless coupling g is expected to be small in typical experiments of cold atoms and therefore Δ_s can be much smaller than nc_1 , which is in contrast with an antiferromagnetic quantum spin-1 system where a spin gap is of the order of an exchange energy [26]. This is due to the fact that although an exchange energy is essentially only one energy scale in such a spin system, g in a spinor gas can be determined by the competition among the three different length scales: $a_2 - a_0, a_\perp, 1/n^{1/3}$. On the other hand, the realization of the strong-coupling regime would be important to obtain $\Delta_s > T$, which ensures the spin-liquid properties discussed above. To this end, optical lattice technique, optical Feshbach resonance [55], or microwave Feshbach resonance [56] to tune the spin-dependent interaction may be promising.

At the same time, even when the weak-coupling regime is realized and $\Delta_s < T$ is met, we have a chance to obtain an interesting regime $\Delta_s < T < nc_0, nc_1$. Due to $T < nc_1$, the usage of the nonlinear σ model to describe the low-energy property of the spin sector is still reasonable. Then, it turns out that the spin sector is described by a semiclassical wave, which has been originally discussed in quantum spin systems [23–26]. In this regime, a decay of a two-point correlation function in spin $\langle m_i(r)m_j(0) \rangle$ can be characterized by a correlation length $\xi \sim \frac{\hbar v_s}{T} \log(T/\Delta_s)$. An important point is that a crossover with respect to different behaviours of the correlator occurs around ξ , that is, the decay can be described by algebraic one for $|r| \lesssim \xi$ while it can be described by exponential one for $r \gg \xi$ [1]. Thus, the properties of this regime can be again captured by means of the correlation function.

C. Species

So far, spin-1 bosons for $c_1 > 0$ have been discussed, which have been realized in ^{23}Na [53]. In this case, the ratio of the spin-dependent coupling to the spin-independent coupling c_1/c_0 is indeed small and is of the order of 10^{-2} , which may open a window of opportunity to see the properties discussed above.

We can also consider ^{87}Rb atoms where the ferromagnetic condensate has been realized in the three dimensional regime since $c_1 < 0$ [57]. In this case, the ratio $|c_1|/c_0$ is of the order of 10^{-3} , which is smaller than the spin-1 ^{23}Na case. At the same time, the change of properties for $c_1 < 0$ may not be drastic compared with that for $c_1 > 0$ since it is known that a ferromagnetic property always maintains even in one dimensional case [58].

In addition, a spin-2 BEC has also been realized with ^{87}Rb [57, 59]. In the spin-2 case, we need an additional case since there are two-independent spin-dependent couplings, c_1 and c_2 each of which describes the spin-spin coupling and pair-singlet coupling, respectively. For a spin-2 ^{87}Rb BEC realized in 3D, the ground state is expected to be a nematic phase, which is realized for $c_1 > 0$ and $c_2 < 0$. The ratios c_1/c_0 and $|c_2|/c_0$ are respectively of the order of 10^{-2} and 10^{-3} . The nematic phase has similar properties as the polar phase in the sense that there is no magnetization and

the spin-singlet amplitude takes a nonzero value in the ground state [39, 60, 61]. However, the nematic phase has an unusual property since it has an accidental $SO(5)$ symmetry at the semiclassical level, which leads to the emergence of quasi-Nambu-Goldstone modes [62]. Thus, such modes gain masses due to the explicit symmetry breaking from the interaction term with $SO(3)$ via the quantum fluctuations even in 3D. When it comes to the reduced dimensional case, however, the spin-singlet projection operator possessing $SO(5)$ symmetry in the spin-2 case are responsible for spin gaps and then, $SO(3)$ symmetric interaction may just cause the renormalization on the gaps [54]. Thus, as in the case of the polar phase in a spin-1 BEC, the excitation spectra except for the charge sector obtain gaps in such a spin-2 Bose gas in 1D.

In addition, the ratios of a spin-dependent coupling to a spin-independent coupling may be changed by using the optical [55] and microwave Feshbach resonances [56].

V. SUMMARY

We have discussed spinor bosons confined by an elongated trap and shown that the system experiences different properties by changing a radial confinement. At absolute zero, we have predicted the spin liquid condensate phase realized in an intermediate regime where the charge sector is 3D while the

spin sector is 1D. In this condensate, the charge $U(1)$ symmetry is spontaneously broken but the spin rotational symmetry is unbroken due to a spin gap. We have pointed out that at a finite temperature higher than the spin gap but lower than a spin-dependent coupling, there is the regime where the spin sector can be described by a semiclassical wave. We suggest that each phase can be characterized by properties of correlation functions and topological defects.

We also comment on a possible experiment to probe the phases discussed above. The simplest way would be to measure the spin gap since each phase can be discussed by existence or non-existence of the gap or its magnitude as shown in the previous sections. Then, magnon contrast interferometer recently demonstrated to measure a magnon gap in Ref. [63], magnetic resonance spectroscopy [64], or Bragg spectroscopy in a spin-selective manner [2] may be utilized to measure the spin gap.

It would also be interesting to extend our study to higher spin cases.

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